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PERIODIC DECIMAL FRACTIONS.

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Although decimal fractions present themselves very early in the mathematical course yet they offer many questions to which it is difficult to find an answer in the text-books or even in the mathematical encyclopedias. Some of these questions are of such an elementary type that their answers can be appreciated even by students of little mathematical training. Moreover, periodic decimals offer "an unlimited amount of material for practice, leading to results whose beauty and easy verification have a great charm."*

The main object of the present paper is to collect the principal theorems relating to periodic decimals and to present them in a form which can easily be understood. It is believed that a useful service may thus be rendered to teachers of secondary mathematics, especially since these theorems are so scattered and relate to such an elementary subject. In addition to this, the writer has proved some of the results by more modern methods, especially by using elementary properties of group theory. These methods of proof are the only elements of novelty claimed for the present article. An outline of these methods was published by Professor Miller, *Bulletin of the American Mathematical Society*, Vol. XIV, 1908, page 356.

In the *Nouvelles Annales de Mathématiques* for 1842 M. Catalan has summarized the principal theorems then known regarding decimals, basing the greater number of his proofs on Fermat's theorem. Another summary by Laisant and Beaujeux which includes several additional theorems appears in the same magazine for 1868. The proofs given in this article are developed from the standpoint of radix fractions, *i. e.*, fractions developed according to the powers of a base. Since only a few of the theorems stated in these two summaries, or elsewhere, are found in the mathematical encyclopaedias or in the texts on arithmetic, algebra, or number theory, it has seemed desirable to present briefly the more important ones in this article. Although no attempt has been made to exhibit new theorems, in several instances, the proofs have been modernized by the application of elementary group theory principles.

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*Weber und Wellstein, *Encyklopaedie der Elementar-Mathematik*, Vol. 1, 1906, p. 258.

Possibly the theorems which are best known and most frequently found in encyclopaedias and texts are the three which will be stated first. Proofs for the first and third of these theorems may be found, for instance, in Chrystal's *Algebra*, second edition, Part I, page 171, while the proof for the second theorem is almost self-evident.

It is assumed in the theorems that will be stated that m/n is a proper fraction in lowest terms.

Theorem I. The necessary and sufficient condition that a decimal terminates is that n must be of the form $2^{\alpha} 5^{\beta}$ where α and β may both be positive integers or either α or β may be zero.

Theorem II. If n contains powers of 2 and 5 as well as other factors, the powers of 2 and 5 may be removed, and after a certain number of places, the fraction will have the same mantissa as some fraction with a denominator prime to 10.

Theorem III. Any rational fraction m/n, n being prime to 10, is periodic.

The general rules which govern the number of places in the decimal period, considering the fraction as having unity for a numerator will now be given, and then it will be shown that the length of this period is independent of the value of the numerator.

Theorem IV. Given any fraction $1/p$, p being an odd prime, finding the period of this fraction consists simply in finding the exponent to which 10 belongs, modulus p.

If 10 is a primitive root of p, the period will be of length $p-1$. If 10 is not a primitive root of p, the exponent will be some divisor of $\phi(p)$, according to Fermat's theorem.

Theorem V. If in the fraction $1/n$, n is the product of different odd primes, the period of $1/n$ is equal to the least common multiple of the periods of the primes into which n can be resolved.

This theorem is proved by the following theorem: If $n=p^{\alpha} q^{\beta} r^{\gamma} \ldots$, where p, q, r, etc., are different primes, and if f, g, h, etc., are the exponents to which a belongs, moduli p^{α}, q^{β}, r^{γ}, ..., then t being the least common multiple of f, g, h, ..., $a^t \equiv 1$, modulus n. When $a=10$, $10^t \equiv 1$, modulus n, or the required period is of length t.

Theorem VI. If n is of the form p^{a} and $\frac{10^s-1}{p}$ is prime to p, s being the exponent to which 10 belongs, modulus p, then the length of the resulting period is equal to the length of the original period multiplied by p^{a-1}.

When 10 is a primitive root of p and also a primitive root of p^2, then 10 belongs to exponent $p-1$, modulus p, and to exponent $\phi(p^2)$ or $p(p-1)$, modulus p^2. From the theory of primitive roots it is known that in order that 10 may be a primitive root of p and also a primitive root of p^2, $\frac{10^{p-1}-1}{p}$ must be prime to p. Since any primitive root of p^2 is also a primitive root of p^α, the theorem holds for p^α. If 10 is not a primitive root of p, 10 must belong to some exponent which is a divisor of $p-1$ and it may be shown that in this case the theorem also holds.

Theorem VII. The length of the period of m/n is independent of the value of the numerator.

If n is any given number, there are always $\phi(n)$ different numbers less than n and prime to n. Then if m/n is any proper fraction in lowest terms, there are always $\phi(n)$ different fractions with a denominator n having numerators less than n and prime to n. But the $\phi(n)$ different numbers less than n and prime to n form a group in respect to multiplication, modulus n. Therefore, the numerators of the given fractions also form a group in respect to multiplication, modulus n.

For every fraction m/n, n prime to 10, 10 will be one of the $\phi(n)$ numbers less than n and prime to n, if $10<n$. If $10>n$, some residue of 10, modulus n, will occur. The powers of 10, modulus n, form a cyclic sub-group G_1 of the larger multiplication group G.

All the fractions having numerators belonging to G_1 will have a period of the same length as $1/n$. Assume that $10^x\equiv 1$, modulus n, or that the period of $1/n$ contains x places. Then $1/n=.a_1a_2...a_xa_1a_2...$ Multiplying by 10 simply changes the decimal point each time one place to the right, or the length of the period for each fraction whose numerator belongs to G_1 is of the same length as that of $1/n$. There is only a cyclical interchange of the numbers composing the period.

An operator of G not in G_1 multiplied by an operator of G_1 will give some distinct element of the group not in the cyclic sub-group. Let k be such an operator in G. Then suppose $k/n=.\beta_1\beta_2\beta_3...\beta_1\beta_2\beta_3...$ Multiplying by 10 simply moves the decimal point one place to the right, therefore, the multiples of k/n by powers of 10 will have a period of the same length as k/n. But it has been assumed that 10 belongs to exponent x, modulus 10 or $10^x\equiv 1$, modulus n, therefore, $k/n\times 10^x$ gives a period of the same length as that of the original period of the cyclic sub-group. If the operators of G are not yet exhausted, another operator not already used may be chosen and the above reasoning repeated. It is then clear, since all fractions with numerators belonging to G_1 have the same period as $1/n$ and all others have the same period as fractions whose numerators belong to G_1 that the length of the period is independent of the value of the the numerator. A concrete example will serve to illustrate these statements. Assume $n=21$. Then $\phi(21)=\phi(3)\phi(7)=12$ and the fractions having numerators prime to 21 are as follows: $\frac{1}{21}$, $\frac{2}{21}$, $\frac{4}{21}$, $\frac{5}{21}$, $\frac{8}{21}$, $\frac{10}{21}$, $\frac{11}{21}$, $\frac{13}{21}$, $\frac{16}{21}$, $\frac{17}{21}$, $\frac{19}{21}$, $\frac{20}{21}$. In this case the cyclic sub-group G_1 of the numerators will be 1, 10, 16, 13, 4, 19. The periods of the fractions having these numbers for numerators will be found by a cyclical interchange of the the numbers .047619. Taking 2 as an operator in G another set of six numbers would be given by the fractions $\frac{2}{21}$, $\frac{20}{21}$, $\frac{11}{21}$, $\frac{5}{21}$, $\frac{8}{21}$, and $\frac{17}{21}$. The periods would be a cyclical interchange of the numbers .095238.

The number of different cyclic sets, or different periods of the same length, for each denominator n is equal to $\phi(n)$ divided by the exponent to which 10 belongs, modulus n. This is true since $\phi(n)$ gives the number of fractions having numerators in the multiplication group G and the exponent to which 10 belongs, modulus n, gives the number of these fractions in each cyclic set. For instance, the number 10 belongs to exponent 6, modulus 21, and $\phi(21)$ is 12. Therefore, there are two different cyclic sets for $n=21$, as was seen in the previous example.

The operation of finding the numbers composing the different cyclic sets of the $\phi(n)$ fractions is very much shortened by the fact, that, in general, the different cyclic sets, or periods, occur as complements of each other. Since -1 and $+1$ always occur among the $\phi(n)$ numbers of G, the $\phi(n)$ fractions whose numerators differ only in respect to sign, modulus n, have periods occurring in complementary pairs. Then if a period has been obtained, its complementary period is found by subtracting each digit of the first period from nine. If 10 is a primitive root of n, then -1 is in the cyclic sub-group G_1. The period is in this case of even length and the second half of the period is obtained by subtracting the digits of the first half from nine. The period is of even length since G_1 then contains an operator of order two and the order of the operator must divide the order of G_1. If -1 is not in G_1, then the index of G_1 under G must be even and the periods are complementary.

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THE DEFLECTING FORCE OF THE EARTH'S ROTATION AND FOUCAULT'S PENDULUM: AN ELEMENTARY ANALYSIS.

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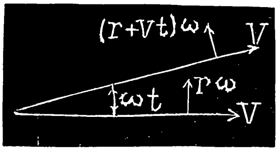
The intrinsic interest in tangible evidences of the Earth's rotation makes it desirable to introduce the following results at as early a stage as is possible. This is the justification for the following elementary exposition.

1. *General Method.* Let a point move about a center with constant angular velocity ω, and recede from it radially with constant velocity v, and let it be initially at a distance r from the center.

After a time t the radial velocity is in a direction making an angle ωt with its initial direction. The component velocities along and perpendicular to the initial radius CP are therefore initially v, $r\omega$. After a time t they are, respectively,



$$v\cos\omega t-(r+vt)\omega\sin\omega t,$$
$$v\sin\omega t+(r+vt)\omega\cos\omega t.$$

To find the components of acceleration, we must